General Economic equilibria: Variational inequalities and existence; Altruism and efficient market outcomes

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Contents

Abstract.

The main goal of the talk is to provide an introduction to the so called General Economic Equilibrium Model.

We also present two research projects in that area. First, we discuss an application of Variational Inequality theory to show existence of equilibria (joint work with Maria Bernadette Donato). Then, we investigate the possibility of modelling altruism as a way to increase efficiency of the equilibrium outcome.

1 The general equilibrium approach to economics

A definition of Economics (or microeconomics or neoclassical economics):

”Economics is the study of the use of scarce resources to satisfy unlimited human wants” (Lipsey and others (1990)).

Central problem in economics: Given an arbitrary nonempty set X, a function $f : X \longrightarrow \mathbb{R}$, a set $C \subseteq X$,

$$\max_{x \in X} f(x) \quad \text{subject to} \quad x \in C.$$

In other words, we want to describe the set

$$\{x^* \in X : x^* \in C \text{ and for any } x \in C,\ f(x^*) \geq f(x)\}.$$

Definition of general equilibrium approach. Object of general equilibrium analysis: markets,

i.e., a set of individuals who own goods and may decide to exchange them in order to improve their well being.

In this talk, we consider an exchange economy: individuals are households or consumers (no production).

Jevons (1871), Menger (1871), Walras (1874).

Arrow and Debreu (1952), McKenzie (1959).

Debreu (1970), Smale (1972).

Below, we present the elements which describe an exchange economy model.

1.1 Set of goods or commodities

We assume there is a number $C \in \mathbb{N}$ of goods or commodities (objects which can be consumed or exchanged).

A commodity is denoted by $c \in \{1, ..., C\} := \mathcal{C}$.

A commodity bundle is a vector in $\mathbb{R}^C$.

The commodity set is a set $X \subseteq \mathbb{R}^C$.

Examples.

1. $X \subseteq \mathbb{R}^C_+$.

$x := (x^c)_{c=1}^C$ is interpreted as the bundle of commodities which, for any $c \in C$, contains x^c units of commodity c.

2. X^T.

$x := \left(\left(x^{it}\right)_{i=1}^{n}\right)_{t=1}^{T}$, $T \in N$.The superscript $t \in \{1, ...T\}$ represents the date in which the commodity is available.

3. $\widetilde{X}^{(1+S)}$, $S \in N$.

$$x := \left(\left(x^{is}\right)_{i=1}^{n}\right)_{s=0}^{S}.$$

$s = 0$ is interpreted as today, $s \geq 1$ is interpreted as a possible "state of the world" which can occur tomorrow.

4. (A subset of the) Space of all sequences of vectors in $\widetilde{X}$.

5. $C^0\left(\mathbb{R}_+, \widetilde{X}\right)$.

$x(t)$ is the vector of commodities at time $t \in \mathbb{R}_+$.

6. A topological vector space.

1.2 Households

Given $H \in \mathbb{N}$, $\mathcal{H} := \{1, ..., H\}$ is the set of households or consumers with generic element h.

The goal of each household $h \in \mathcal{H}$ is to choose an element in the commodity set.

Her choices

are based on her preferences over a subset X_h of the commodity set, called her consumption set.

are limited by the economic environment to a subset of the consumption set called her budget set.

In the present version of the model, (exchange economy), the budget set depends upon

the element of X owned by the household, her endowment, and

the exchange ratios at which commodities are exchanged, i.e., commodity prices.

1.3 Consumption set of household h

The consumption set of household h is the subset X_h of X the household can conceivably consume, given her specific physical and institutional constraints imposed by the environment.

An element $x_h \in X_h$ is called consumption bundle and it can be interpreted as the element in X_h consumed by household h. $(x_h)_{h \in \mathcal{H}}$ is called consumption allocation.

Examples.

1. $\forall h \in \mathcal{H},\ X_h = X$.

2. $\left\{x_h \in \mathbb{R}^n_+ : x^1_h \geq 1\right\}$.

1.4 Preferences

Fundamental characteristics of a household are her “tastes” or “preferences”.

Preferences may be described by a set valued function or a binary relation or an utility function.

1. A binary relation $\succeq_h$ on the consumption set X_h (called a preference relation).

$x_h \succeq_h y_h$ is interpreted as “at least as good as” or “weakly preferred to” by household h.

2. A set valued function

$$P_h : X_h \longrightarrow\!\!\!\!\longrightarrow X_h, \qquad x_h \mapsto\!\!\!\mapsto P_h(x_h)$$

(a set valued function from X to X is a function from X to $P(X)$, the family of all subsets of X).

$P_h(x_h)$ is interpreted as the set of all elements in the choice set X_h which are strictly preferred to x_h.

3. A function $u_h : X_h \longrightarrow \mathbb{R}$, $x_h \mapsto u_h(x_h)$ is called an utility function.

$u_h(x_h) \geq u_h(y_h)$ is interpreted as "at least as good as" or "weakly preferred to" by household h.

Several properties could be imposed on P_h or $\succeq_h$ or u_h. For example,

Pr1 $\succeq_h$ is complete, i.e., $\forall x, y \in X_h$, $x \succeq_h y$ or $y \succeq_h x$.

Pr2 $\succeq_h$ is transitive, i.e.,

$\forall x, y, z \in X_h$, if $x \succeq_h y$ and $y \succeq_h z$, then $x \succeq_h z$.

Pr3 $\succeq_h$ is continuous,i.e., $\forall x \in X_h$, $\{x' \in X_h : x' \succeq_h x\}$ and $\{x' \in X_h : x \succeq_h x'\}$ are closed sets.

Definition. A function $u_h : X_h \to \mathbb{R}$ is a utility function representing the preference relation $\succeq_h$ if

$$\forall x_h, y_h \in X_h,\ x_h \succeq_h y_h \Leftrightarrow u_h(x_h) \geq u_h(y_h)$$

Theorem 1 *Assume that* $\succeq_h$ *defined on* $\mathbb{R}^n_+$ *is complete, transitive, and continuous. Then, there exists a continuous utility function* u_h *representing* $\succeq_h$.

For any $h \in H$, let $\mathcal{S}_h, \mathcal{R}_h, \mathcal{U}_h$ be the set of set valued preference functions, preference relations and utility functions, respectively, we decided to use to describe preferences of each household. Define

$$P = (P_h)_{h\in\mathcal{H}} \in (\times_h \mathcal{S}_h) := \mathcal{S},$$

$$u = (u_h)_{h\in\mathcal{H}} \in (\times_h \mathcal{U}_h) := \mathcal{U},$$

$$\succeq = (\succeq_h)_{h\in\mathcal{H}} (\times_h \mathcal{R}_h) := \mathcal{R}.$$

Let $(t, \mathcal{T})$ denote the chosen way to describe preferences.

1.5 Endowments

The endowment set E_h is the subset of the commodity set consisting of all

commodity bundles a household may own.

Denote an element of E_h by e_h .

$e := (e_h)_{h \in \mathcal{H}} \in (\times_h E_h) := E$.

Examples.

1. $\forall h \in \mathcal{H},\ E_h = X$.

2. $E_h = \mathbb{R}^n_+ \setminus \{0\}$.

1.6 Economies

Definition. An economy is a pair of

1. an endowment,

2. either a preference or set-valued preference or a utility function,

i.e., $(e, P) \in E \times \mathcal{S}$, or

$(e, \succeq) \in E \times \mathcal{R}$, or

$(e, u) \in E \times \mathcal{U}$.

1.7 Prices

A price p is a function $p : X \to \mathbb{R}$, with $p(x)$ interpreted as the value of x, or the number of units of "money" (gold) which has to be paid to get x or which can be obtained selling x.

The price set is denoted by P.

Example.

$p : \mathbb{R}^C \to \mathbb{R}_+$, $p(x) = \overline{p}x = \sum_{c=1}^{C} \overline{p}^c x^c$,

where $\overline{p}^c$ is the price of one unit of commodity c and $\overline{p} \in \mathbb{R}^C$.

In this case, we can identify P with $\mathbb{R}^C$.

1.8 Budget sets

The budget set of household h is the subset of the consumption set X_h which contains elements the household can afford to buy given the economic constraints defining the economy under analysis. In our case,

$$\beta(p, e_h) = \{x_h \in X_h : p(x_h) \leq p(e_h)\}$$

1.9 Households behavior

Each household h solves the following problem: (Ph)

Given $t \in \mathcal{T}$, $e_h \in E_h, p \in P$,

$$\text{Find } \overline{x}_h \in X_h \text{ such that}$$

$$\overline{x}_h \in \beta(p, e_h) \quad \text{and}$$

$$P_h(\overline{x}_h) \cap \beta(p, e_h) = \varnothing,$$

or

$$\text{Find } \overline{x}_h \in X_h \text{ such that}$$

$$\overline{x}_h \in \beta(p, e_h) \quad \text{and}$$

$$\text{for any } y_h \in \beta(p, e_h), \quad \overline{x}_h \succeq_h y_h,$$

or

$$\max_{x_h \in X_h} \quad u_h(x_h) \qquad \text{subject to} \qquad x_h \in \beta(p, e_h),$$

1.10 Definition of equilibria

Summarizing, we have described the following basic concepts of our model:

1. Commodity set X,

2. Set H of households. Each of them described by

   (a) a consumption set X_h,

   (b) preferences;

(c) an endowment e_h belonging to the endowment set E_h;

3. Price set P.

We call equilibrium a situation in which prices are such that the total consumption of each good is smaller or equal than the total resources of that good, i.e., the sum of each household's endowments.

$(x_h)_{h\in\mathcal{H}}$ is called an allocation (of goods).

Definiition. $((x_h)_{h\in\mathcal{H}}, p) \in (\times_{h\in\mathcal{H}} X_h) \times P$ is an equilibrium allocation-price for an economy $(e,t) \in E \times \mathcal{T}$ if

1. households maximize, i.e., for $h \in \mathcal{H}$, x_h solves problem (Ph) at (e_h, t_h, p);

2. $(x_h)_{h\in\mathcal{H}}$ satisfies the following (market clearing) conditions

$$\sum_{h\in\mathcal{H}} x_h \leq \sum_{h\in\mathcal{H}} e_h \tag{1}$$

Definition 1 *Let* $X(e,t)$ *be the set of equilibrium allocations associated with an economy* $(e,t) \in \mathcal{S} \times \mathcal{T}$*.*

Theorem 2 *Under some assumptions, for any economy an equilibrium exists - see below.*

Theorem 3 *Under some (weaker) assumptions, for any* $(e, t) \in E \times \mathcal{T}$, *any* $(x_h)_{h \in \mathcal{H}} \in X(e, t)$ *is*

efficient (or Pareto Optimal),

i.e., $\forall \left(x'_h\right)_{h \in \mathcal{H}} \in \times_{h \in \mathcal{H}} X_h$ *such that* $\sum_h x'_h = \sum_h x_h$,

there exists $h^* \in \mathcal{H}$ *such that* $u_{h^*}\left(x'_{h^*}\right) < u_{h^*}(x_{h^*})$

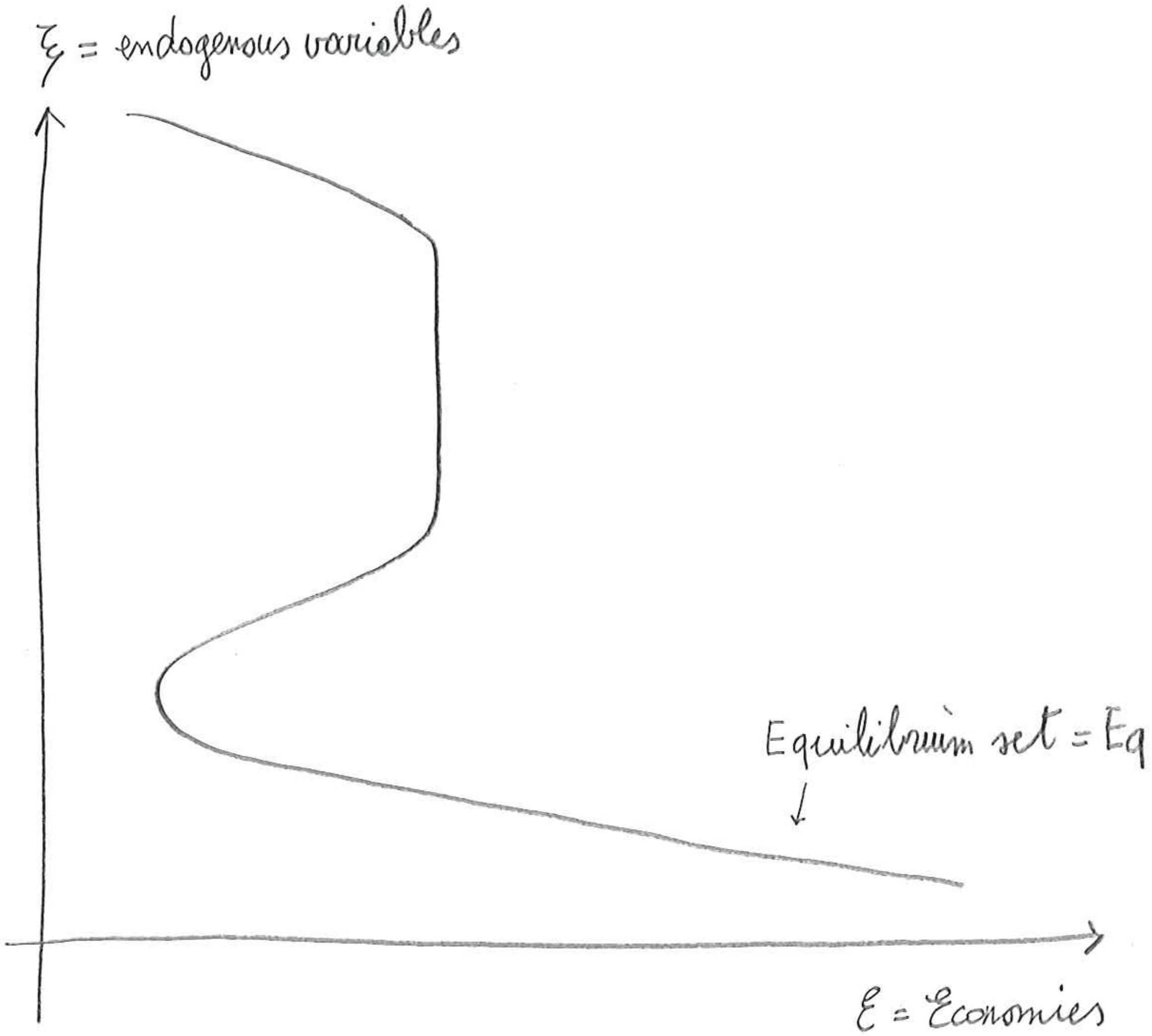
(any redistribution is vetoed by at least one household).

Theorem 4 *For any* $u \in \mathcal{U}$ *and under some smoothness assumptions on utility functions,*

there exists an open and full Lebesque measure set $\mathcal{F}$ *contained in* $\times_h E_h = \mathbb{R}_{++}^{CH}$ *such that*

$\forall e \in \mathcal{F}$, $\# X(e, u)$ *is finite and*

the equilibrium is locally a smooth function of the endowments.



$$Eq = \{(\xi, \mathcal{E}) : \xi \text{ is an equilibrium associated with } \mathcal{E}\}$$

Mathematical tools:

Existence: fixed point, homotopy, variational inequalities methods.

Pareto Optimality: simple argument.

Generic regularity. Basic differential topology.

2 Variational Inequality problems and general equilibrium

2.1 Variational inequality problems

Fichera (1964), Lions and Stampacchia (1965).

Definition 2 *Given*

a nonempty, closed and convex subset C *of* $\mathbb{R}^n$ *and*

the set-valued maps $S : C \to P(\mathbb{R}^n)$ *and* $\Phi : C \to P(\mathbb{R}^n)$.

A Generalized Quasi-Variational Inequality problem associated with C, S, Φ, *denoted by GQVI, is the following problem:*

$$\begin{array}{l} \text{“Find } \widetilde{x} \in S(\widetilde{x}),\ \varphi \in \Phi(\widetilde{x}) \text{ such that} \\ \text{for any } x \in S(\widetilde{x}),\ \langle \varphi, x - \widetilde{x} \rangle_n \geq 0\ \text{.”} \end{array} \tag{2}$$

In particular, when $S(x) = C$ for all $x \in C$, (2) is a Generalized Variational Inequality (GVI); when Φ is single-valued, (2) reduces to the Quasi-Variational Inequality (QVI). Finally, when both Φ is single-valued and $S(x) = C$, for all $x \in C$, we have the classical Variational Inequality (VI).

Definition 3 *Let us consider a nonempty, closed and convex subset C of $\mathbb{R}^n$.*

(GQVI) Let us consider the set-valued maps $S : C \to 2^{\mathbb{R}^n}$ and $\Phi : C \to 2^{\mathbb{R}^n}$. A Generalized Quasi-Variational Inequality associated with C, S, Φ is the following problem:

"Find $\widetilde{x} \in S(\widetilde{x})\,,\ \varphi \in \Phi(\widetilde{x})$

such that $\langle \varphi, x - \widetilde{x} \rangle_n \geq 0 \qquad \forall x \in S(\widetilde{x})$."

Definition 4 *(GVI) Let us consider the set-valued map* $\Phi : C \to 2^{\mathbb{R}^n}$. *A Generalized Variational Inequality associated with* C, Φ *is the following problem:*

"Find $\widetilde{x} \in C$, $\varphi \in \Phi(\widetilde{x})$ *such that*

$$\langle \varphi, x - \widetilde{x} \rangle_n \geq 0 \qquad \forall x \in C \ ."$$

Definition 5 *(QVI) Let us consider the set-valued map* $S : C \to 2^{\mathbb{R}^n}$ *and the function* $\phi : C \to \mathbb{R}^n$. *A Quasi-Variational Inequality associated with* C, S, ϕ *is the following problem:*

"Find $\widetilde{x} \in S(\widetilde{x})$, *such that*

$$\langle \phi(\widetilde{x}), x - \widetilde{x} \rangle_n \geq 0 \qquad \forall x \in S(\widetilde{x}) \ ."$$

(VI) Let us consider the function

$$\phi : C \to \mathbb{R}^n.$$

The (Classical) Variational Inequality associated with ϕ is the following problem:

"Find $\widetilde{x} \in C$ such that

for any $x \in C$, $\qquad \langle \phi(\widetilde{x}), x - \widetilde{x} \rangle_n \leq 0\,.$"

Relationship between solutions to the above inequality problems and equilibria in the exchange economy model.

1. Households' maximization problem.

Assume that u is C^1 and concave, then for given $p \in \Delta := \left\{p' \in \mathbb{R}^n_+ : \sum_{i=1}^n p'^i = 1\right\}$,

$$\widetilde{x}_h \text{ solves the consumer problem } (Ph),$$

$\Leftrightarrow$

$$\widetilde{x}_h \in \beta_h(p, e_h) \text{ and it is such that}$$

$$\forall x_h \in \beta_h(p, e_h), \; Du(\widetilde{x}_h) \cdot (x_h - \widetilde{x}_h) \leq 0.$$

Intuition: if the utility function is concave, a point in which all directional derivatives are negative is a global maximum.

2. Market clearing conditions.

Let $\widetilde{x}_h$ be a solution to household's h maximization problem for any $h \in H$.

$$
\begin{array}{l}
\sum_{h\in\mathcal{H}} (\widetilde{x}_h - e_h) \leq 0. \\
\Leftrightarrow \\
\widetilde{p} \in \mathbf{\Delta} \text{ is such that } \forall p \in \mathbf{\Delta}, \\
\left(\sum_{h\in\mathcal{H}} (\widetilde{x}_h - e_h)\right) \cdot (p - \widetilde{p}) \leq 0,
\end{array} \tag{3}
$$

Intuition: Under monotonicity assumptions of the utility functions, for any $h \in H$, $\widetilde{x}_h$ solves the household maximization problem implies

$\left(\sum_{h\in\mathcal{H}} (\widetilde{x}_h - e_h)\right) \widetilde{p} = 0$, and then, from the variational inequality in (3), we get

$$
\forall p \in \mathbf{\Delta}, \quad \left(\sum_{h\in\mathcal{H}} (\widetilde{x}_h - e_h)\right) \cdot p \leq 0.
$$

Choosing $p = (0, .., 0, 1, 0, ..., 0) \in \Delta$, with 1 in the $c - th$ position, you get $\left(\sum_{h \in \mathcal{H}} \left(\widetilde{x}_h^c - e_h^c\right)\right) \leq 0$.

2.2 A result on general equilibrium using a VI approach

Goal: A general results on existence of equilibrium prices in a general equilibrium using the VI approach.

The variational inequality literature has presented several results with respect to

a. maximization problems;

b. general economic equilibrium and Nash equilibrium;

The economic literature has presented much more general results about those problems using several versions of fixed point theorems.

We try to get results of the same level of generality of the economic literature using the VI approach: we are working on it ...

Assumptions. For any $h \in \mathcal{H}$,

(i) X_h is non-empty, closed, convex and bounded below;

(ii) P_h is lower semicontinuous; (P_h is lsc at $x_h \in X_h$ if $P_h(x_h) \neq \varnothing$ and for any open set V in X such that $P_h(x_h) \cap V \neq \varnothing$, there exists an open neighborhood U of x_h such that for every $x'_h \in U$, $P_h\left(x'_h\right) \cap V \neq \varnothing$);

(iii) (Openness like assumption) For any $x_h \in X_h$, $y_h \in P_h(x_h)$ and $z_h \in X_h \backslash \{y_h\}$, we have $[z_h, y_h) \cap P_h(x_h) \neq \varnothing$.

(iv) P_h is convex valued.

(v) (Irreflexivity) For any $x_h \in X_h$, $x_h \notin P_h(x_h)$.

(vi) (Global Nonsatiation) For any $x_h \in X_h$, there exists $y_h \in X_h$ such that $y_h \in P_h(x_h)$.

(vi*) For any $x_h \in X_h$, there exists $y_h \in P_h(x_h)$ such that for any $z_h \in \mathbb{R}^C \setminus \{y_h\}$, $[z_h, y_h) \cap P(x_h) \neq \varnothing$;

(vii) $e_h \in \text{Int}_{\mathbb{R}^C}(X_h)$.

Given a preference relation $\succsim_h$, then the associated set value preferences P_h is

$$P_h : X_h \longrightarrow\longrightarrow X_h,\ P_h(x_h) = \{y_h \in X_h : y_h \succ_h x_h\}.$$

If u_h is upper semicontinuous at x_h and x_h is not a global maximum for u_h, then the associated P_h is lsc at x_h.

Theorem 5 *Under the above Assumptions, for any economy an equilibrium exists.*

Remark. The above result assume neither completeness nor transitivity of preferences. It is also more general than the result which assumes continuous, quasi-concave, locally nonsatiated utility functions.

Comparison with VI literature on general economic equilibrium models.

To the best of our knowledge, the most general result on existence of equilibria is the one contained in (Donato, Milasi and Vitanza (2018)). There, existence is proved in terms of utility functions (and not general preferences as in the case of the set-valued function P). The needed assumptions there are (i), (v), (vi), (vi), (vii) and also the fact that the utility function is continuous and semi strictly quasiconcave. Continuity is strictly stronger than Assumptions (ii), (iii) and (vi*), and with semistrict quasi concavity is strictly stronger than quasiconcavity.

Comparison with economic literature.

To the best of our knowledge the most general result is contained in Gourdel (1995).

Proposition 6 *An equilibrium exists if the following Assumptions are satisfied. For any $h \in H$,*

(i) *X_h is non-empty, closed, convex and bounded below;*

(ii) *P_h is lower semicontinuous;*

(iii) *(Openness like assumption) For any $x_h \in X_h$, $y_h \in P_h(x_h)$ and $z_h \in X_h \setminus \{y_h\}$, we have $[z_h, y_h) \cap P_h(x_h) \neq \varnothing$;*

(iv') *For any $x_h \in X_h$, $x_h \notin \text{conv}(P_h(x_h))$.*

(vi) *(Global Nonsatiation) For any $x_h \in X_h$, there exists $y_h \in X_h$ such that $y_h \in P_h(x_h)$.*

(vii) $e_h \in \mathsf{Int}_{\mathbb{R}^C}(X_h)$.

The above result is strictly more general than ours because it does not assume (vii*) and Assumptions (iv) and (v) imply (iv'), and not viceversa.

3 General equilibrium, Altruism and efficiency of markets

3.1 General motivation

Economic Right-wing viewpoint: The free market (or competitive market, or capitalist economy) works well.

Economic Left-wing viewpoint: The free market does not work well; an outside the market institution ("state") can do better.

We use the general equilibrium model to make the above statements precise and then discuss them.

We can say that the market works well if for each economy an equilibrium exists and is efficient.

Thus, the standard general equilibrium model (or Walrasian model) says that the market works well.

The standard Walrasian model holds a right-wing viewpoint.

We now want to argue that there are microeconomic models whose results are in agreement with the “left-wing point of view”.

The standard Walrasian model assumes:

rationality of individuals,

completeness of the information available to them,

total freedom of exchange between goods, or "complete markets",

absence of strategic interaction between the individuals, and therefore absence of phenomena such as externalities, public goods, information asymmetries, market structures with a "small" number of agents, the possibility of forming coalitions .... .

The violation of the simplifying assumptions above have been termed "market imperfections" - even if their absence can be considered an imperfection of the model itself.

In (many of) the above defined models with market imperfections, it has been shown that

for almost all economies, an equilibrium exists, but all associated equilibria are inefficient.

For many economists, the above statement is not enough to say that the free market works badly, simply because an intervention from the outside would make the system work “worse”.

What does “better” or “worse”?

One allocation is better than another one if it is Pareto superior to the other, that is, if all consumers are better off in the first one than in the second one.

In many of the models with so-called “market imperfections”, it has been shown that

for almost all economies,

a subject external to the market who has considerable information, has no gain or loss from different market outcomes (has no “conflicts of interest”) and has sufficient power can

lead to a superior Pareto equilibrium allocation than that existing one before the intervention,

without eliminating the imperfection of the market, and

with small size intervention.

Reasons to criticize the intervention of an external subject must be based on lack of information, or presence of conflicts of interest or lack of power.

3.2 Pareto Optimality, altruism, state intervention (in progress)

(Work in progress)

We analyze an exchange economy general equilibrium model. The only differences with respect to the standard cases are what follows:

1. each household utility function is a linear combination of a “her own selfish utility function” and other households welfare;

2. households are allowed to do transfers to other households.

Kranich (1988).

We want to show that there exists a type of planner intervention which may coexist with Pareto improvement and altruistic behavior of households.

More formally, households are allowed to transfer goods to other households:

$t^c_{h,h'} \in \mathbb{R}_+$ is the transfer of good $c \in \mathcal{C}$ from household h to household h',

$$t_{h,h'} := \left(t^c_{h,h'}\right)_{c \in \mathcal{C}} \in \mathbb{R}^C_+,$$
$$t_h := \left(t_{h,h'}\right)_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{C(H-1)}_+,$$
$$t := (t_h)_{h \in \mathcal{H}} \in \mathbb{R}^{C(H-1)H}_+ := T$$
$$t_{\setminus h} := (t_{h'})_{h' \in \mathcal{H} \setminus \{h\}} \in \mathbb{R}^{C(H-1)(H-1)}_+$$

Moreover, we assume that the objective function w_h of each household h is a combination of the "selfish" utility function and of a function of transfers made to other households as follows.

$$w_h : \mathbb{R}^{CH}_{++} \times \mathbb{R}^{C(H-1)} \to \mathbb{R},$$

$$(x_h, t_h) \mapsto \; u_h(x_h) + \textstyle\sum_{h' \neq h} \alpha_{hh'} \cdot v_{hh'}\left(t_{h,h'}\right),$$

Then an economy is $(u, v, e, \alpha) \in \mathcal{U} \times \mathcal{V} \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^{H(H-1)}$.

Definition 6 *The vector* $(\widetilde{x}, \widetilde{t}, \widetilde{p}) \in \mathbb{R}^{CH}_{++} \times \mathcal{T} \times \mathbb{R}^{C}_{++}$ *is an equilibrium vector for the economy* $(u, v, e, \alpha) \in \mathcal{U} \times \mathcal{V} \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^{H(H-1)}$ *if*

1. **Definition 7** *(a) for any* $h \in \mathcal{H}$*,*

   for given $(u, v, e, \alpha, \widetilde{p}, \widetilde{t}_{\setminus h}) \in \mathcal{U} \times \mathcal{V} \times \mathbb{R}^{CH}_{++} \times \mathbb{R}^{H(H-1)} \times \mathbb{R}^{C}_{++} \times \mathbb{R}^{C(H-1)(H-1)}$,

   $(\widetilde{x}_h, \widetilde{t}_h) \in \mathbb{R}^{CH}_{++} \times \mathbb{R}^{C(H-1)}$ *solves problem*

$$\max_{(x_h, t_h)} \quad u_h(x_h) + \sum_{h' \neq h} \alpha_{hh'} \cdot v_{h,h'}\left(t_{h,h'}\right)$$

subject to

$$px_h + p\sum_{h' \neq h} t_{h,h'} \leq p \cdot e_h + p\sum_{h' \neq h} t_{h',h}$$

$$t_{h,h'} \geq 0. \tag{4}$$

(b) markets clear, i.e.,

$$\sum_{h \in \mathcal{H}} \widetilde{x}_h \leq \sum_{h \in \mathcal{H}} e_h.$$

Proposition 7 *For any economy an equilibrium exists.*

Definition. A household $h \in \mathcal{H}$ is potentially altruistic with rispect to household $h' \in \mathcal{H}$ if $\alpha_{hh'} > 0$.

Proposition 8 *If the number of potentially altruistic household is large enough, for any* $(u, v) \in \mathcal{U} \times \mathcal{V}$*, there exists a non empty, open subsets* Σ^* *of the space of endowments for which*

for all associated equilibria there exist an increase in altruism and a choice of taxes on transfers which is Pareto improving.

Basic mathematical tools: differential topology (Sard Theorem, homotopy analysis).

1. We start our analysis from a function whose zeros describe equilibria:

$$F : \Xi \times \Theta \to \mathbb{R}^n,\ F : (\xi, \theta) \mapsto F(\xi, \theta)$$

where Ξ, an open subset of R^n, is the set of endogenous variables ξ and Θ is the set of the exogenous variables.

2. We define a new equilibrium function

$$\widetilde{F} : \Xi \times \mathbb{R}^T \times \Theta, \quad (\xi, \tau, \theta) \mapsto \widetilde{F}(\xi, \tau, \theta),$$

taking into account the planner's intervention effects on agents behaviors via some policy tools $\tau \in R^T$.

3. We observe that there is a value $\overline{\tau}$ at which equilibria with and without planner's intervention coincide.

4. We define a goal function

$$G : \Xi \times \mathbb{R}^T \times \Theta, \quad (\xi, \tau, \theta) \mapsto G(\xi, \tau, \theta)$$

and we analyze the local effect of a change in τ around $\overline{\tau}$ on G when its arguments assume their equilibrium values.

To accomplish the analysis described in 4., we proceed through the following steps.

(a) Applying the Implicit Function Theorem, we show that there exist a neighborhood N of $\overline{\tau}$ and a unique C^1 function h defined on N, such that

$$\text{for } \tau \in N, \qquad \widetilde{F}(h(\tau), \tau, \theta) = 0$$

The function h describes how equilibrium variables and dependent tools adjust to changes in planner's independent tools τ_1. Then the function

$$g_\theta : N \to \mathbb{R}^k, \ \tau \mapsto G\left(h\left(\tau\right), \tau, \theta\right)$$

describes how the goal function changes when the planner uses her policy tools τ and variables move in the equilibrium set defined by $\widetilde{F}$.

(b) The objective of the analysis is to show that there exists an open and dense subset $S^* \subseteq \Theta$ such that for each $\theta \in S^*$, the planner can "move" the equilibrium value of the goal function in any directions locally around $g_\theta\left(\overline{\tau}\right)$, the value of the goal function in the case of no intervention.

In other words, for any direction of movement away from $g_\theta(\overline{\tau})$ and for any neighborhood N of $\overline{\tau}$, there exists a point $\tau^* \in N$ such that $g_\theta(\tau^*)$ belong to that directions.

(c) A sufficient condition for g_θ to typically satisfy the above described condition is that

$$\text{rank } [Dg(\overline{\tau})]_{k \times T} = k$$

Some technical fact

Theorem 9 (Sard's theorem in euclidean spaces) *Let U be an open subset of R^m and $f : U \to R^n$ be a C^r function, with*

$$r > \max\{m - n, 0\}$$

Then, the set CP_f of critical values of f has measure zero in R^n.

Definition 8 *An element $x \in M$ is a regular point for f if df_x is surjective, and it is a critical point for f otherwise. An element $y \in N$ is a regular value for f if every x in $f^{-1}(y)$ is regular, and it is a critical value otherwise.*

Remark 1 *Notice that $y \in N$ is regular if $y \notin f(M)$; in particular, when $m < n$ $y \in N$ is a regular value if and only if $y \notin f(M)$. On the other hand, $y \in N$ is a critical value if and only if it is the image of a critical point.*

The set of all the regular points for f and the set of all the critical points for f are called RP_f and CP_f respectively; the set of all the regular values for f and the set of all the critical values for f are called R_f and C_f respectively.

Remark 2 *The equality* $C_f = f\left(CP_f\right)$ *is always true, whereas the equality* $R_f = f\left(RP_f\right)$ *is generally false: first, because any* $y \notin \operatorname{Im} f$ *is regular; second, because* $f^{-1}(y)$ *may contain regular as well as critical points. Thus* $R_f \cap \operatorname{Im} f = f\left(RP_f\right)$ *if and only if, for each* y *in* N*, the elements of* $f^{-1}(y)$ *are either all regular or all critical.*

Definition 9 *Let X be a topological space. $f : X \longrightarrow \mathbb{R}$ is lsc if for any $\alpha \in \mathbb{R}$,*

$\{x \in X : f(x) > \alpha\}$ *is X-open.*

Proposition. $f : X \longrightarrow \mathbb{R}$ is lsc $\Leftrightarrow$

$\varphi_1 : X \longrightarrow\longrightarrow \mathbb{R}$, $\varphi_1(x) = (-\infty, f(x)]$ is lsc $\Leftrightarrow$

$\varphi_2 : X \longrightarrow\longrightarrow \mathbb{R}$, $\varphi_1(x) = [f(x, +\infty))$ is usc.