Concorso per l'ammissione al corso di Dottorato di Ricerca in Matematica, Informatica, Statistica

Ciclo XXXI

Curriculum in Mathematics

Examination 2

Applicants are required to do some of the following exercises

1. Let the *Chebyshev's polynomials of the first kind* be

   $$T_0(x) \equiv 1, \quad T_1(x) = x, \qquad T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x), \quad k = 1, 2, \ldots.$$

   Prove that the polynomials

   $$\hat{T}_0(x) = T_0(x), \qquad \hat{T}_k(x) = 2^{1-k}T_k(x), \quad k = 1, 2, \ldots,$$

   are monic. Moreover, prove that $\hat{T}_k(x)$ is the monic polynomial of degree k of least norm on the interval $[-1, 1]$.

   Explain in details the importance of such polynomials in the framework of polynomial interpolation.

2. Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix. Define its LU factorization and state under which conditions it exists.

   Prove that if A is symmetric and positive definite, then it is always factorizable LU.

3. Briefly discuss the notion of a quotient space $X/\sim$ of a topological space X by an equivalence relation $\sim$.

   (a) Prove that for the natural projection p of X onto $X/\sim$ the following holds: if $X/\sim$ is connected and $p^{-1}(y)$ is connected for every y in $X/\sim$, then X is connected.

   (b) Prove that the natural projection from $\mathbb{C}^n$ onto the quotient group $\mathbb{C}^n/\mathbb{Q}^n$ (equipped with the quotient topology) is open.

   (c) Determine whether $\mathbb{C}/\mathbb{Q}$ is T_1 and whether it is compact.

4. Briefly discuss the linear isometries of $\mathbb{R}^n$ with respect the the Euclidean metric. In $\mathbb{R}^n$ let d the distance defined as follows:

   $$d((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = \#\{i = 1, \ldots, n \mid x_i \neq y_i\}.$$

   (a) Prove that d is in fact a distance.

   (b) Prove that for a linear subspace V of $\mathbb{R}^n$

   $$\min\{d(v, w) \mid v, w \in V, v \neq w\} \leq n - \dim V + 1, \tag{1}$$

   and for each k with $1 \leq k \leq n$ provide an example of a linear subspace V of $\mathbb{R}^n$ with dimension k for which equality holds in (1).

   (c) Determine the linear isometries of $(\mathbb{R}^n, d)$.

5. Briefly discuss Sylow theory for finite groups. For a prime p, a finite group is said to be p-closed if it has a unique Sylow p-subgroup.

   (a) Prove that each group of order 20 is 5-closed.

   (b) Prove that each group of order 30 is both 5-closed and 3-closed.

   (c) Prove that each group of order 30 has a cyclic subgroup of index 2 and deduce that there exist at most 4 groups of order 30 (up to isomorphism).

6. Briefly discuss the notion of an affine algebraic set.

   (a) Let $\mathbb{K}$ be a field and let V be an affine algebraic set in $\mathbb{K}^n$. Prove that the ring of polynomial functions on V is a domain if and only if V is irreducible.

   (b) Determine whether the affine algebraic set $V(< x^4+y^4-y(x^2+y^2) >)$ in $\mathbb{C}^2$ is irreducible.

   (c) Determine whether the affine algebraic set in (b) is rational and if this is the case describe a parametrization explicitly. (Recall that for a non-constant polynomial $F \in \mathbb{C}[x,y]$ the affine algebraic set $V(<F>)$ in $\mathbb{C}^2$ is said to be *rational* if there exist $f(t), g(t), h(t), l(t) \in \mathbb{C}[t]$, with $g \neq 0$ and $l \neq 0$, such that $(f(z)/g(z), h(z)/l(z)) \in V(<F>)$ for each $z \in \mathbb{C}$ with $g(z)l(z) \neq 0$, and also the corresponding map is non-constant. Such a map is termed a parametrization of $V(<F>)$).

7. State a theorem about pointwise, uniform or quadratic convergence of Fourier series.

   Given the function
   $$f(x) = e^{x^2} \text{ for } x \in [-\pi, \pi)$$
   periodically defined in $\mathbb{R}$ (with period 2π) and denoted by a_n and b_n its Fourier coefficients, prove that

   (a)
   $$\lim_{n\to+\infty} na_n = \lim_{n\to+\infty} nb_n = 0;$$

   (b)
   $$\sum_{n=1}^{+\infty} na_n \sin\left(\frac{n\pi}{2}\right) = -\pi e^{\frac{\pi^2}{4}}.$$

8. State the maximum principle for harmonic functions. Denote by D the disk $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ and consider $u \in C^2(D) \cap C^0(\overline{D})$ solution of the Dirichlet problem
   $$\begin{cases} \Delta u &= 0 & \text{in } D, \\ u(x,y) &= x^2y^2 & \text{on } \partial D. \end{cases}$$

   (a) Evaluate
   $$\max_{\overline{D}} u \quad \text{and} \quad \min_{\overline{D}} u.$$

(b) Prove that

$$\frac{\partial u}{\partial y}(x,0) = 0 \text{ for every } x \in (-1,1).$$

9. Two two mass points, P_1 e P_2, both of mass m, are constrained (without frictions) to move on a fixed circle of radius R, contained in a vertical plane, and are linked by a spring of elastic constant k and zero length at rest (Figura 1). By choosing the Lagrangian coordinates φ and ψ as in figure,

    (a) write down the Lagrange equations of second kind for the system;

    (b) determine the equilibrium configurations and study their stability;

    (c) determine the frequencies of the small oscillations close to the stable equilibrium configuration.

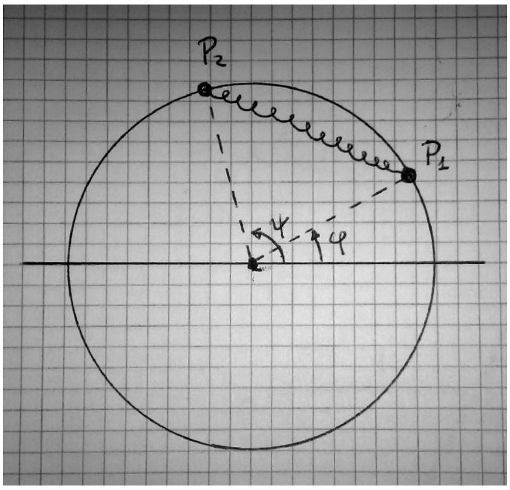

Figura 1:

10. The central point Q of a homogeneous rod of mass M and length $2L$ is constrained to move (without frictions) on a rectilinear horizontal guide which rotates, with constant angular velocity ω, around a vertical axis containing its fixed point O. The rod is free to rotate (without friction) around Q in such a way that the rod is always contained in the vertical plane determined by the guide and the vertical direction (Figura 2). The point Q is linked to O by a spring of elastic constant k and zero length at rest. Finally, at one endpoint of the rod a mass point P of mass m is fixed. By choosing the Lagrangian coordinates φ and q as in figure,

    (a) write down the Lagrange equations of second kind for the system;

    (b) assuming $m = 0$, determine the equilibrium configurations of the rod, and study their (relative) stability.

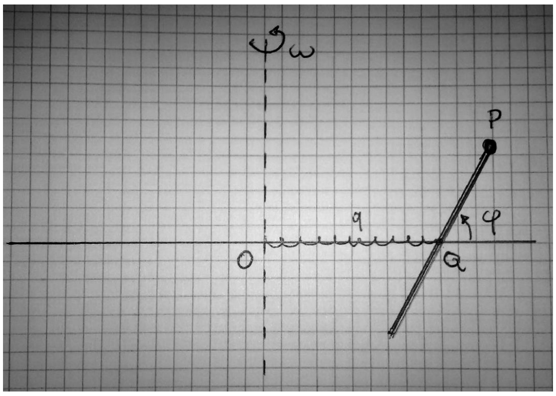

Figura 2:

11. Introduce the concept of random vector, joint distribution, marginal distribution and measures of centrality and dispersion relating to it.

    Let a two-dimensional random vector uniformly distributed in a circle about the origin of radius 2 be given. Then

    (a) determine the marginal density functions $f_X(x)$ and $f_Y(y)$;

    (b) compute $\mathbb{P}(x/y = 1)$;

    (c) compute $\mathbb{P}(|x| \leq 1/y = 1)$.

12. Introduce the concept of function of a random vector, and describe (and prove) at least one significant result.

    Let X be a random variable uniformly distributed in $[-\frac{\pi}{2}, \frac{\pi}{2}]$; then:

    (a) Compute $E(X)$ and $Var(X)$;

    (b) If $Y = \tan X$ compute the density probability of Y and study $E(Y)$ and $Var(Y)$.